**Factoring - Everything you NEED to know**

**Topic 1: Greatest Common Factors and Factoring by Grouping**

*Factoring* is the opposite of multiplying; it is the process of expressing a polynomial as a product of two or more polynomials.

For example, the factors of $10x^2 + 15x$ are $5x$ and $(2x + 3)$, that is $10x^2 + 15x = 5x \cdot (2x + 3)$.

*Prime polynomials* are polynomials that can't be factored using integer coefficients.

**Finding the Greatest Common Factor (GCF)**

The greatest common factor is an expression of the highest degree that divides each term of the polynomial. Notice that this expression contains the lowest power of the common variable.

For example, to find the GCF of $x^3y^2$, $xy^4$, $x^4y^5$ we first find the lowest power for the variable $x$, which is $x^1$ the lowest power that appears in all three terms, and then find the lowest power that appears for variable $y$, which is $y^2$ the lowest power that appears in all three terms, so the GCF for this example is $xy^2$.

**Example:** Find the GCF of the following: $6(x - 4)^2$, $3(x - 4)^3$, $18(x - 4)^4$.

**Solution:**

The three numbers 6, 3, 18 have 3 as their greatest common factor. The lowest power of $(x - 4)$ in any of the three terms is $(x - 4)^2$. Therefore, the GCF is $3(x - 4)^2$. 
Factoring a Monomial from a Polynomial

1. Determine the GCF of all terms in the polynomial.
2. Write each term as the product of the GCF and another factor.
3. Use the distributive property to factor out the GCF.

Example: Factoring out the GCF

Factor $15x^4 - 5x^3 + 20x^2$

Solution:
1. Determine the GCF: $5x^2$
2. Express each term as the product of the GCF and another factor:
   $5x^2 \cdot 3x^2 - 5x^2 \cdot x + 5x^2 \cdot 4$
3. Use the distributive property to factor out the GCF: $5x^2 \cdot (3x^2 - x + 4)$

Example: Factoring out the GCF

Factor $20x^3y^3 + 6x^2y^4 - 12xy^5$

Solution:
1. GCF: $2xy^3$
2. Express each term as the product of the GCF and another factor:
   $20x^3y^3 + 6x^2y^4 - 12xy^5 = 2xy^3 \cdot 10x^2 - 2xy^3 \cdot 3xy + 2xy^3 \cdot 6y^2$
3. Use the distributive property to factor out the GCF:
   $2xy^3 \cdot (10x^2 + 3xy - 6y^2)$

Example: Using Common Factors with a Negative Coefficient

Factor $-2x^3 + 6x^2 - 18x$

Solution:
1. GCF: $-2x$
2. Express each term as the product of the GCF and another factor:
   $-2x^3 + 6x^2 - 18x = -2x \cdot (x^2) - 2x \cdot (-3x) - 2 \cdot (6)$
3. Use the distributive property to factor out the GCF: $-2x \cdot (x^2 - 3x - 6)$
Example: Factoring Out the Greatest Common Binomial Factor

Factor $3x(5x - 2) + 4(5x - 2)$

Solution:
1. GCF: $5x - 2$
2. Express each term as the product of the GCF and another factor:
   
   $3x \cdot (5x - 2) + 4 \cdot (5x - 2)$

3. Use the distributive property to factor out the GCF: $(5x - 2) \cdot (3x + 4)$

Example: Factoring Out the Greatest Common Binomial Factor

Factor $9(2x - 5) + 6(2x - 5)^2$

Solution:
1. GCF: $3(2x - 5)$
2. Express each term as the product of the GCF and another factor:
   
   $9(2x - 5) + 6(2x - 5)^2 = 3 \cdot 3(2x - 5) + 3(2x - 5) \cdot 2(2x - 5)$

3. Use the distributive property to factor out the GCF and simplify:
   
   $3 \cdot 3(2x - 5) + 3(2x - 5) \cdot 2(2x - 5) = 3(2x - 5) \cdot (3 + 2(2x - 5))$
   
   $= 3(2x - 5) \cdot (3 + 4x - 10)$
   
   $= 3(2x - 5) \cdot (4x - 7)$
Factoring by Grouping

When a polynomial has four terms, it may be possible to factor by grouping.

Factoring by Grouping
1. Arrange the four terms into two groups of two terms each. Each group of two terms must have a GCF
2. Factor the GCF from each group of two terms.
3. If the two terms formed in step 2 have a GCF, then factor it out.

Example: Factoring by Grouping

Factor \(x^3 - 5x^2 + 2x - 10\)

Solution:
1. Arrange the four terms: \((x^3 - 5x^2) + (2x - 10)\)
2. Factor the GCF from each group of two terms:
   \((x^3 - 5x^2) + (2x - 10) = 5x^2(x - 5) + 2(x - 5)\)
3. Factor common terms:
   \(x^2(x - 5) + 2(x - 5) = (x - 5)(x^2 + 2)\)

Example: Factoring by Grouping

Factor \(x^4 - 5x^3 + 2x^2 - 10x\)

Solution:
1. Arrange the four terms: \((x^4 - 5x^3) + (2x^2 - 10x)\)
2. Factor the GCF from each group of two terms:
   \((x^4 - 5x^3) + (2x^2 - 10x) = x^3(x - 5) + 2x(x - 5)\)
3. Factor common terms:
   \(x^3(x - 5) + 2x(x - 5) = (x^3 + 2x)(x - 5) = x(x^2 + 2)(x - 5)\)
**Topic 2: Factoring Trinomials**

In this section, we will see how to factor trinomials of the form: 

\[ ax^2 + bx + c, \ a \neq 0 \]

**Factoring Trinomials of the Form:** \( ax^2 + bx + c, \ a = 1 \) 

**Sum and Product Method**

1. Find two numbers (or factors) whose product is \( c \) and whose sum is \( b \).
2. The factors of the trinomial will be of the form: 

   \((x + \square)(x + \square)\) 
   
   The first box takes the value of the first factor from step 1 and the second box takes the value of the other factor from step 1.

**Example: Factoring a Trinomial Whose Leading Coefficient is 1** 

**Factor:** \( x^2 + 6x + 8 \)

**Solution:**

1. Find two numbers whose product is 8 and whose sum is 6. 
   List all possible factors of 8 and then add their factors:

   \[
   \begin{array}{cc}
   \text{Product} & \text{Sum} \\
   1 \times 8 & 1 + 8 = 9 \\
   2 \times 4 & 2 + 4 = 6 \\
   \end{array}
   \]

2. \( x^2 + 6x + 8 = (x + 2)(x + 4) \)

**Example: Factoring a Trinomial Whose Leading Coefficient is 1** 

**Factor:** \( x^2 - 10x + 9 \)

**Solution:**

1. Find two numbers whose product is 9 and whose sum is -10. 
   List all possible factors of 9 and then add their factors:

   \[
   \begin{array}{cc}
   \text{Product} & \text{Sum} \\
   (-1) \times (-9) & (-1) + (-9) = -10 \\
   (-3) \times (-3) & (-3) + (-3) = -6 \\
   \end{array}
   \]

2. \( x^2 - 10x + 9 = (x - 1)(x - 9) \)
Example: Factoring a Trinomial Whose Leading Coefficient is 1

Factor: $x^2 + 7x - 60$

Solution:
1. Find two numbers whose product is -60 and whose sum is 7. List all possible factors of -60 and then add their factors:

<table>
<thead>
<tr>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1)\cdot 60 = -60)</td>
<td>((-1) + 60 = 59)</td>
</tr>
<tr>
<td>(1\cdot (-60) = -60)</td>
<td>(1 + (-60) = -59)</td>
</tr>
<tr>
<td>((-2)\cdot 30 = -60)</td>
<td>((-2) + 30 = 27)</td>
</tr>
<tr>
<td>(2\cdot (-30) = -60)</td>
<td>(2 + (-30) = -27)</td>
</tr>
<tr>
<td>((-3)\cdot 20 = -60)</td>
<td>((-3) + 20 = 17)</td>
</tr>
<tr>
<td>(3\cdot (-20) = -60)</td>
<td>(3 + (-20) = -17)</td>
</tr>
<tr>
<td>((-4)\cdot 15 = -60)</td>
<td>((-4) + 15 = 11)</td>
</tr>
<tr>
<td>(4\cdot (-15) = -60)</td>
<td>(4 + (-15) = -11)</td>
</tr>
<tr>
<td>((-5)\cdot 12 = -60)</td>
<td>((-5) + 12 = 7)</td>
</tr>
<tr>
<td>(5\cdot (-12) = -60)</td>
<td>(5 + (-12) = -7)</td>
</tr>
<tr>
<td>((-6)\cdot 10 = -60)</td>
<td>((-6) + 10 = 4)</td>
</tr>
<tr>
<td>(6\cdot (-10) = -60)</td>
<td>(6 + (-10) = -4)</td>
</tr>
</tbody>
</table>

2. $x^2 + 7x - 60 = (x - 5)(x + 12)$

Example: Factoring a Trinomial Whose Leading Coefficient is 1

Factor: $x^2 - 4x - 21$

Solution:
1. Find two numbers whose product is -21 and whose sum is -4. List all possible factors of -21 and then add their factors:

<table>
<thead>
<tr>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1)\cdot 21 = -21)</td>
<td>((-1) + 21 = 20)</td>
</tr>
<tr>
<td>(1\cdot (-21) = -21)</td>
<td>(1 + (-21) = -20)</td>
</tr>
<tr>
<td>((-3)\cdot 7 = -21)</td>
<td>((-3) + 7 = 4)</td>
</tr>
<tr>
<td>(3\cdot (-7) = -21)</td>
<td>(3 + (-7) = -4)</td>
</tr>
</tbody>
</table>

2. $x^2 - 4x - 21 = (x + 3)(x - 7)$
Factoring Trinomials of the Form: $ax^2 + bx + c$, $a \neq 1$
(Complex Trinomials)

Decomposition method

1. Find two numbers (or factors) whose product is $a \cdot c$ and whose sum is $b$.
2. Rewrite (decompose) the middle term $bx$ using the numbers found in step 1.
3. Factor by grouping.

Example:
Factor: $2x^2 - 5x - 12$

Solution:
Here, $a = 2$, $b = -5$, and $c = -12$. We need to find two numbers whose product is $a \cdot c = 2 \cdot (-12) = -24$ and whose sum is $b = -5$.

<table>
<thead>
<tr>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-1) \cdot 24 = -24$</td>
<td>$(-1) + 24 = 23$</td>
</tr>
<tr>
<td>$1 \cdot (-24) = -24$</td>
<td>$1 + (-24) = -23$</td>
</tr>
<tr>
<td>$(-2) \cdot 12 = -24$</td>
<td>$(-2) + 12 = 10$</td>
</tr>
<tr>
<td>$2 \cdot (-12) = -24$</td>
<td>$2 + (-12) = -10$</td>
</tr>
<tr>
<td>$(-3) \cdot 8 = -24$</td>
<td>$(-3) + 8 = 5$</td>
</tr>
<tr>
<td>$3 \cdot (-8) = -24$</td>
<td>$3 + (-8) = -5$</td>
</tr>
<tr>
<td>$(-4) \cdot 6 = -24$</td>
<td>$(-4) + 6 = 2$</td>
</tr>
<tr>
<td>$4 \cdot (-6) = -24$</td>
<td>$4 + (-6) = -2$</td>
</tr>
</tbody>
</table>

Rewrite $-5x$ as $-8x + 3x$, so now we have $2x^2 - 8x + 3x - 12$
Performing the last step,

\[
2x^2 - 8x + 3x - 12 = 2x(x - 4) + 3(x - 4)
= (2x + 3)(x - 4)
\]
**Example:**
Factor: \(3x^2 - 3x - 6\)

**Solution:**
Here, \(a = 3\), \(b = -3\), and \(c = -6\). We need to find two numbers whose product is \(a \cdot c = 3 \cdot (-6) = -18\) and whose sum is \(b = -3\).

<table>
<thead>
<tr>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1) \cdot 18 = -18)</td>
<td>((-1) + 18 = 17)</td>
</tr>
<tr>
<td>(1 \cdot (-18) = -18)</td>
<td>(1 + (-18) = -17)</td>
</tr>
<tr>
<td>((-2) \cdot 9 = -18)</td>
<td>((-2) + 9 = 7)</td>
</tr>
<tr>
<td>(2 \cdot (-9) = -18)</td>
<td>(2 + (-9) = -7)</td>
</tr>
<tr>
<td>((-3) \cdot 6 = -18)</td>
<td>((-3) + 6 = 3)</td>
</tr>
<tr>
<td>(3 \cdot (-6) = -18)</td>
<td>(3 + (-6) = -3)</td>
</tr>
</tbody>
</table>

Rewrite -3x with 3x - 6x, so now we have \(3x^2 - 3x - 6 = 3x^2 + 3x - 6x - 6\)
Performing the last step,
\[
3x^2 + 3x - 6x - 6 = 3x(x + 1) - 6(x + 1)
= (3x - 6)(x + 1)
\]
**Topic 3: Special Factoring**

*Factoring the Difference of Two Squares (DoTS):*

\[ a^2 - b^2 = (a + b)(a - b) \]

How to recognize difference of two squares:
1. Two terms
2. Subtraction
3. Both terms can be square rooted (after factoring out the GCF)

How to factor it:
1. Take out the GCF
2. Find the square root of each of the two terms
3. Write their sum in one bracket... difference in the other

**Example:**
Factor: \( x^2 - 25 \)

**Solution:**
Here \( \sqrt{x^2} = x \), and \( \sqrt{25} = 5 \).
Therefore: \( x^2 - 25 = (x + 5)(x - 5) \)

**Example:**
Factor: \( 27w^2 - 48v^2 \)

**Solution:**
\[
27w^2 - 48v^2 = 3(9w^2 - 16v^2) \]
Here \( \sqrt{9w^2} = 3w \), and \( \sqrt{16v^2} = 4v \).
\[
= 3(3w + 4v)(3w - 4v)
\]

**Example:**
Factor: \( x^4 - 16 \)

**Solution:**
\[
x^4 - 16 = (x^2 + 4)(x^2 - 4) \]
difference of squares again
\[
= (x^2 + 4)(x + 2)(x - 2)
\]
**Perfect Square Trinomials:**

Sometimes a trinomial factors into a product of two identical brackets
This is called a perfect square trinomial

How to recognize a perfect square trinomial:
1. Three terms
2. First and last terms are perfect squares (can be square rooted)
3. Middle term is *twice the product of the square roots* of the first and last terms

**Example:**
Factor: \( x^2 - 20x + 25 \)

*Check: The last term must be positive too*

**Solution:**
Note: \( 20x \) is the product of \( x \) and 5, doubled (ignore the sign on the 20 for now)

\[
x^2 - 20x + 25 = (x - 10)^2
\]

**Example:**
Factor: \( 16x^2 + 56x + 49 \)

**Solution:**
Note: \( 56x \) is the product of \( 4x \) and 7, doubled (ignore the sign on the 20 for now)

\[
16x^2 + 56x + 49 = (4x + 7)^2
\]