Dividing Polynomials by Polynomials

In this section, you will learn about

- Dividing polynomials by polynomials
- Writing powers in descending order
- Missing terms

**Introduction.** In this section, we will conclude our discussion on operations with polynomials by discussing how to divide one polynomial by another.

Dividing polynomials by polynomials

To divide one polynomial by another, we use a method similar to long division in arithmetic. We illustrate the method with several examples.

**Example 1** **Dividing polynomials.** Divide \( x^2 + 5x + 6 \) by \( x + 2 \).

**Solution**

Here the divisor is \( x + 2 \), and the dividend is \( x^2 + 5x + 6 \).

**Step 1:**

\[
\frac{x}{x + 2 \mid x^2 + 5x + 6} \quad \text{How many times does} \ x \ \text{divide} \ x^2? \ x^2 \div x = x.
\]

Place the \( x \) above the division symbol.

**Step 2:**

\[
\frac{x + 2}{x + 2 \mid x^2 + 5x + 6} \quad \text{Multiply each term in the divisor by} \ x. \ \text{Place the product under} \ x^2 + 5x \ \text{and draw a line.}
\]

\[
x^2 + 2x
\]

**Step 3:**

\[
\frac{x}{x + 2 \mid x^2 + 5x + 6} \quad \text{Subtract} \ x^2 + 2x \ \text{from} \ x^2 + 5x \ \text{by adding the negative (opposite) of} \ x^2 + 2x \ \text{to} \ x^2 + 5x.
\]

\[
\frac{x^2 + 2x}{3x + 6} \quad \text{Bring down the 6.}
\]

**Self Check**

Divide \( x^2 + 7x + 12 \) by \( x + 3 \).
**Example 1**

Dividing polynomials. Divide \( \frac{6x^2 - 7x - 2}{2x - 1} \).

**Solution**

Here the divisor is \( 2x - 1 \) and the dividend is \( 6x^2 - 7x - 2 \).

**Step 1:**

\[
\frac{3x}{2x - 1} \frac{6x^2 - 7x - 2}{6x^2} \]

How many times does \( 2x \) divide \( 6x^2 \)? \( 6x^2 + 2x = 3x \).
Place the \( 3x \) above the division symbol.

**Step 2:**

\[
\frac{3x}{2x - 1} \frac{6x^2 - 7x - 2}{6x^2} \]

Multiply each term in the divisor by \( 3x \). Place the product under \( 6x^2 - 7x \) and draw a line.

**Step 3:**

\[
\frac{3x}{2x - 1} \frac{6x^2 - 7x - 2}{6x^2} \]

Subtract \( 6x^2 - 3x \) from \( 6x^2 - 7x \) by adding the negative (opposite) of \( 6x^2 - 3x \) to \( 6x^2 - 7x \).  
\[
-4x - 2 \]

Bring down the \(-2\).

**Step 4:**

\[
\frac{3x - 2}{2x - 1} \frac{6x^2 - 7x - 2}{6x^2} \]

How many times does \( 2x \) divide \(-4x\)? \(-4x + 2x = -2\).  
Place the \(-2\) above the division symbol.
Step 5: Multiply each term in the divisor by \(-2\). Place the product under \(-4x - 2\) and draw a line.

\[
\begin{array}{c}
3x - 2 \\
6x^2 - 3x \\
-4x - 2 \\
-4x + 2
\end{array}
\]

Step 6: Subtract \(-4x + 2\) from \(-4x - 2\) by adding the negative (opposite) of \(-4x + 2\).

\[
\begin{array}{c}
3x - 2 \\
6x^2 - 3x \\
-4 \\
-4
\end{array}
\]

Here the quotient is \(3x - 2\) and the remainder is \(-4\). It is common to write the answer in this form:

\[
3x - 2 + \frac{-4}{2x - 1}
\]

Quotient + \frac{\text{remainder}}{\text{divisor}}.

Step 7: To check the answer, we multiply

\[
3x - 2 + \frac{-4}{2x - 1} \quad \text{by} \quad 2x - 1
\]

The product should be the dividend.

\[
(2x - 1) \left(3x - 2 + \frac{-4}{2x - 1}\right) = (2x - 1)(3x - 2) + (2x - 1)\left(\frac{-4}{2x - 1}\right)
\]

\[
= (2x - 1)(3x - 2) - 4
\]

\[
= 6x^2 - 4x - 3x + 2 - 4
\]

\[
= 6x^2 - 7x - 2
\]

Answer: \(4x - 3 + \frac{6}{2x + 3}\)

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**Writing powers in descending order**

The division method works best when the exponents of the terms in the divisor and the dividend are written in descending order. This means that the term involving the highest power of \(x\) appears first, the term involving the second-highest power of \(x\) appears second, and so on. For example, the terms in

\[3x^3 + 2x^2 - 7x + 5\]

have their exponents written in descending order.

If the powers in the dividend or divisor are not in descending order, we use the commutative property of addition to write them that way.

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**EXAMPLE 3** Dividing polynomials. Divide \(4x^2 + 2x^3 + 12 - 2x\) by \(x + 3\).

**Solution**

We write the dividend so that the exponents are in descending order.
\[
\begin{align*}
\frac{2x^2 - 2x + 4}{x + 3} & = \frac{2x^3 + 4x^2 - 2x + 12}{2x^3 + 6x^2} \\
& \quad \frac{-2x^2 - 2x}{2x^3 + 6x^2} \\
& \quad \frac{-2x^2 - 6x}{2x^3 + 6x^2} \\
& \quad \frac{4x + 12}{2x^3 + 6x^2} \\
& \quad \frac{4x + 12}{2x^3 + 6x^2} \\
\end{align*}
\]

Check:
\[
(x + 3)(2x^2 - 2x + 4) = 2x^3 - 2x^2 + 4x + 6x^2 - 6x + 12
\]
\[
= 2x^3 + 4x^2 - 2x + 12
\]

Answer: \(3x^2 + 2x - 4\)

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**Missing terms**

When we write the terms of a dividend in descending powers of \(x\), we must determine whether some powers of \(x\) are missing. For example, in the dividend of

\[
x + 1 \div 3x^4 - 7x^2 + 3x + 15
\]

the term involving \(x^3\) is missing. When this happens, we should either write the term with a coefficient of 0 or leave a blank space for it. In this case, we would write the dividend as

\[
3x^4 + 0x^3 - 7x^2 + 3x + 15 \quad \text{or} \quad 3x^4 - 7x^2 + 3x + 15
\]

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**EXAMPLE 4**  
**Dividing polynomials.** Divide \(\frac{x^2 - 4}{x + 2}\).

**Solution**

Since \(x^2 - 4\) does not have a term involving \(x\), we must either include the term 0\(x\) or leave a blank space for it.

\[
\begin{align*}
\frac{x - 2}{x + 2} & \quad \frac{x^2 - 4}{x + 2} \\
\frac{-2x - 4}{-2x - 4}
\end{align*}
\]

Check:
\[
(x + 2)(x - 2) = x^2 - 2x + 2x - 4
\]
\[
= x^2 - 4
\]

**Self Check**  
Divide \(\frac{x^2 - 9}{x - 3}\).

**Answer:** \(x + 3\)
CONCEPTS In Exercises 5–8, write each polynomial with the powers in descending order.

5. $4x^3 + 7x - 2x^2 + 6$
6. $5x^2 + 7x^3 - 3x - 9$
7. $9x + 2x^2 - x^3 + 6x^4$
8. $7x^5 + x^3 - x^2 + 2x^4$

In Exercises 9 and 10, identify the missing terms in each polynomial.

9. $5x^4 + 2x^2 - 1$
10. $-3x^5 - 2x^3 + 4x - 6$

In Exercises 11 and 12, without doing the division, determine which of the three possible quotients seems reasonable.

11. \[
\frac{x^2 - 81}{x - 3} \quad x^2 + 3x + 9 \\
\quad \quad x^3 + 3x^2 + 9x + 27 \\
\quad \quad x^4 + 3x^3 + 9x^2 + 27x + 1
\]
12. \[
\frac{8x^3 - 27}{2x - 3} \quad 4x^2 + 6x + 9 \\
\quad \quad 4x^3 - 6x^2 - 9 \\
\quad \quad 4x^4 - 6x^3 - 9x^2 + 7
\]

13. a. Solve $d = rt$ for $r$.
   b. Use your answer to part a and the long division method to complete the chart.

<table>
<thead>
<tr>
<th>Subway</th>
<th>$x + 4$</th>
<th>$x^2 + x - 12$</th>
</tr>
</thead>
</table>

14. a. Solve $I = Prt$ for $P$.
   b. Use your answer to part a and the long division method to complete the chart.

<table>
<thead>
<tr>
<th>Bonds</th>
<th>$x + 4$</th>
<th>1</th>
<th>$x^2 + 7x + 12$</th>
</tr>
</thead>
</table>

15. Using long division, a student found that
\[
\frac{3x^2 + 8x + 4}{3x + 2} = x + 2
\]
Use multiplication to see whether the result is correct.

16. Using long division, a student found that
\[
\frac{x^2 + 4x - 21}{x - 3} = x - 7
\]
Use multiplication to see whether the result is correct.

NOTATION In Exercises 17 and 18, complete each division.

17. \[
\begin{array}{c|cc}
X & 2 & +
\end{array}
\]
\[
\begin{array}{c|cc}
2x + 1 & x + 2x^2 + 4x + 4 \\
\hline
x + 2 & 2x + 4 \\
\hline
& 0
\end{array}
\]

18. \[
\begin{array}{c|cc}
X & 6
\end{array}
\]
\[
\begin{array}{c|cc}
2x + 1 & x^2 + 3x^2 - 3x + 5 \\
\hline
x - 2 & 2x^2 - 3x \\
\hline
& 2x^2 + 1 \\
\hline
& -4x + 5 \\
\hline
& 7
\end{array}
\]

Practice In Exercises 19–26, do each division.

19. Divide $x^2 + 4x - 12$ by $x - 2$.


23. \[
\frac{6a^2 + 5a - 6}{2a + 3}
\]

25. \[
\frac{3b^2 + 11b + 6}{3b + 2}
\]

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5. Data circled ones

In Exercises 27–34, write the terms so that the powers of x are in descending order. Then do each division.

27. \(5x + 3\) \(11x + 10x^2 + 3\)
29. \(4 + 2x\) \(-10x - 28 + 2x^2\)
31. \(2x - 1\) \(x - 2 + 6x^2\)
33. \(3 + x\) \(2x^2 - 3 + 5x\)
28. \(2x - 7\) \(-x - 21 + 2x^2\)
30. \(1 + 3\) \(9x^2 + 1 + 6x\)
32. \(2 + x\) \(3x + 2x^2 - 2\)
34. \(x - 3\) \(2x^2 - 3 - 5x\)

In Exercises 35–40, do each division.

35. \(2x + 3\) \(2x^3 + 7x^2 + 4x - 3\)
37. \(3x + 2\) \(6x^2 + 10x^2 + 7x + 2\)
39. \(2x + 1\) \(2x^3 + 3x^2 + 3x + 1\)
36. \(2x - 1\) \(2x^3 - 3x^2 + 5x - 2\)
38. \(4x + 3\) \(4x^3 - 5x^2 - 2x + 3\)
40. \(3x - 2\) \(6x^3 - x^2 + 4x - 4\)

In Exercises 41–50, do each division. If there is a remainder, write the answer in quotient plus remainder form.

41. \(\frac{2x^2 + 5x + 2}{2x + 3}\)
42. \(\frac{3x^2 - 8x + 3}{3x - 2}\)
43. \(\frac{4x^2 + 6x - 1}{2x + 1}\)
44. \(\frac{6x^2 - 11x + 2}{3x - 1}\)

45. \(\frac{x^3 + 3x^2 + 3x + 1}{x + 1}\)
46. \(\frac{x^3 + 6x^2 + 12x + 8}{x + 2}\)
47. \(\frac{2x^3 + 7x^2 + 4x + 3}{2x + 3}\)
48. \(\frac{6x^2 + x^2 + 2x + 1}{3x - 1}\)

49. \(\frac{2x^3 + 4x^2 - 2x + 3}{x - 2}\)

50. \(\frac{3y^3 - 4y^2 + 2y + 3}{y + 3}\)

In Exercises 51–60, do each division.

51. \(\frac{x^2 - 1}{x - 1}\)
52. \(\frac{x^2 - 9}{x + 3}\)
55. \(\frac{x^2 + 1}{x + 1}\)
56. \(\frac{x^2 - 8}{x - 2}\)
53. \(\frac{4x^2 - 9}{2x + 3}\)
54. \(\frac{25x^2 - 16}{5x - 4}\)
57. \(\frac{a^3 + a}{a + 3}\)
58. \(\frac{y^3 - 50}{y - 5}\)
59. \(3x - 4\) \(15x^3 - 23x^2 + 16x\)
60. \(2y + 3\) \(21y^2 + 6y^3 - 2\)

Applications

61. Furnace Filter The area of the furnace filter shown in Illustration 1 is \((x^2 - 2x - 24)\) square inches.
   a. Find its length.
   b. Find its perimeter.

62. Shelf Space The formula \(V = Bh\) gives the volume of a cylinder where \(B\) is the area of the base and \(h\) is the height. Find the amount of shelf space that the container of potato chips shown in Illustration 2 occupies if its volume is \((2x^3 - 4x - 2)\) cubic inches.

Illustration 1

Illustration 2